§ 6. Hecke operators

We've seen before that for $\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n$ Ramanujan conjectured, and Mordel proved that

$$\tau(n)\tau(m) = \tau(nm) \quad \text{if } (n,m)=1 \quad \text{and}$$

$$\tau(p^r)\tau(p) = \tau(p^{r+1}) + p^{11}\tau(p^{r-1})$$

Hecke who defined certain operators $T(n)$ acting on M_k and showed that they form a family of commuting operators that are Hermitian wrt Petersson inner product.

This implies using Linear algebra that the space M_k has a basis consisting of simultaneous eigenforms $\forall T(n)$ and the eigenvalues of these forms inherit the multiplicative properties of the Hecke operators that act on them.

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General setup

Let $G \leq \Gamma = SL_2(\mathbb{Z})$, and

$G \subset K$, where K a set of linear fractional transformations (not necessarily a gp) with the following properties

① G acts on K by multiplication

② $KG = GK = K$

③ $[K:G] < \infty$ i.e. $K = \bigcup_{j=1}^{m} G M_j$, $M_j \in K$

Take $G = \Gamma = SL_2(\mathbb{Z})$

$$K = \mathcal{M}(n) := \{ M \in M_{2\times 2}(\mathbb{Z}) \mid \det M = n \}$$

Clearly G acts on K on the right and on the left since if $\gamma \in \Gamma$, $g \in \mathcal{M}(n)$ then $\det(\gamma g) = (\det \gamma)(\det g) = \det g = \det(g\gamma) = n$.

The next lemma gives an explicit set of reps and also proves $[\mathcal{M}(n) : \Gamma] < \infty$.

<u>Lemma 6-1</u> A ~~set of~~ right coset reps for the action of $PSL_2(\mathbb{Z})$ on the set $\mathcal{M}_n$ of of linear fractional transformations represented by the set of matrices $M(n)$ is given by

$$\mathcal{R} = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \;\middle|\; \begin{matrix} ad = n, & d > 0 \\ 0 \le b < d & \end{matrix} \right\}$$

ie $\mathcal{M}_n = \bigcup_{\gamma_i \in \mathcal{R}} PSL_2(\mathbb{Z})\, \gamma_i$, $\quad \mathcal{R} = PSL_2 \backslash M_n = SL_2 \backslash M(n)$

Proof. Let $\pm \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{M}(n)$ be given

Choose $x, y \in \mathbb{Z}$ such that $Ax + Cy = 0$ and $(x, y) = 1$ (For example take $x = -C$, $y = A$ and divide by $\gcd(A, C)$)

Since $(x,y)=1$ $\exists$ $u,v\in\mathbb{Z}$ s.t $uy-vx=1$

ie $\begin{pmatrix} u & v \\ x & y \end{pmatrix} \in SL_2(\mathbb{Z})$

Then $\begin{pmatrix} u & v \\ x & y \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A' & B' \\ 0 & D' \end{pmatrix}$

Possibly by multiplying on the left by $-I$ we can assume $A'>0$, $D'>0$ -
(Note they have the same sign as $A'D'>0$

Now choose integer q, r with
$B' = D'q + r$, $0 \le r < D'$

Then $\begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A' & B' \\ 0 & D' \end{pmatrix} = \begin{pmatrix} A' & r \\ 0 & D' \end{pmatrix} \in R.$

since $A'D'=n$, $0 \le r < D$, $D'>0$.

Now suppose 2 elts $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, $\begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} \in R$

ared in the same coset ie $\exists$ $\begin{pmatrix} u & v \\ x & y \end{pmatrix} \in SL_2(\mathbb{Z})$

s.t $\begin{pmatrix} u & v \\ x & y \end{pmatrix}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix}$

Then $ax + 0y = 0 \Rightarrow x=0$ (since $a \neq 0$)

Since $\det\begin{pmatrix} u & v \\ x & y \end{pmatrix} = uy - vx = uy = 1 \Rightarrow u = y = 1$ (or -1)
but $d, d'>0 \Rightarrow y=1$

But then $ub + vd = b'$ and $xb + yd = d'$
$\Rightarrow$ $b + vd = b'$ and $b0 + d = d'$
Hence, $b \equiv b' \bmod d$ and $d = d'$

But both $b, b' < d$. Hence
$b = b'$ as well since $d = d'$ and $ad = a'd'$
we also have $a = a'$. ▨.

We now define the Hecke operators

Defn The n-th Hecke operator
$T_k(n) = T_n$) that act on the space of modular forms of wt k is defined by

$$T_k(n) f := n^{k/2-1} \sum_{\gamma_i \in \Gamma \backslash M_n} f|_k \gamma_i$$

where $f|_k \gamma := (\det \gamma)^{k/2} (cz+d)^{-k} f(\gamma z)$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(n)$.

Remark ① This defn is indep of choice of representatives. For, if $\{\gamma_i'\}$ is another set of reps then $\gamma_i' = \sigma_i \gamma_i$ for some $\sigma_i \in \Gamma$.

Then $\sum_i f|\gamma_i' = \sum_i f|\sigma_i \gamma_i = \sum_i f|\gamma_i$

$f|\sigma_i = f$ since $f \in M_k(\Gamma)$

② If $g = T_k(n) f$ and $f \in M_k(\Gamma)$ then so $g|\sigma = g \quad \forall \sigma \in \Gamma$

For, if $\sigma \in \Gamma$ then

$$g|\sigma = (T_k(n) f)|\sigma$$
$$= \left(n^{k/2-1} \sum_i f|\gamma_i\right)|\sigma$$
$$= n^{k/2-1} \sum_i f|\gamma_i\sigma = n^{k/2-1} \sum_i f|\gamma_i'$$
$$= f|T_k(n) = g$$

Since as γ_i runs over a set of reps $\Gamma\backslash M(n)$ so does $\gamma_i' = \gamma_i\sigma$ for a fixed $\sigma \in \Gamma$.

③ We'll see shortly that in fact $g \in M_k(\Gamma)$

Ie $T_k(n) : M_k(\Gamma) \longrightarrow M_k(\Gamma)$

(ie we show that $T_k(n)$ also preserves the holomorphicity at ∞.

Using the explicit coset reps from Lemma 6.1 we will give the action of $T_k(n)$ on the Fourier coefs of f.

Prop 6.2 Let $f(z) = \sum_{n=0}^{\infty} c(n) q^n \in \mathcal{M}_k(\Gamma)$

and $g = T_k(m) f$. Then $g(z) = \sum_{n=0}^{\infty} b(n) q^n$

with $b(n) = \begin{cases} \sum_{d \mid (m,n)} d^{k-1} c\left(\frac{mn}{d^2}\right) & \text{if } n \geq 1 \\ c(0)\, \sigma_{k-1}(m) & \text{if } n = 0 \end{cases}$

Proof: $T_k(m) f = m^{k/2-1} \sum_{\substack{ad=m \\ b \bmod d \\ d>0}} f \Big| \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$

$$= m^{k/2-1} \sum_{a,b,d} d^{-k} f\left(\frac{az+b}{d}\right) m^{k/2}$$

$$= m^{k-1} \sum_{a,b,d} d^{-k} \sum_{n=0}^{\infty} c(n)\, e^{2\pi i n \left(\frac{az+b}{d}\right)}$$

$$= m^{k-1} \sum_{n=0}^{\infty} c(n) \sum_{\substack{a,d \\ ad=m}} d^{-k} e^{2\pi i n \left(\frac{az}{d}\right)} \sum_{b=0}^{d-1} e^{2\pi i n \left(\frac{b}{d}\right)}$$

But $\sum_{b=0}^{d-1} e^{2\pi i n \left(\frac{b}{d}\right)} = \begin{cases} d & \text{if } d \mid n \\ 0 & \text{otherwise} \end{cases}$

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$$\left(\begin{array}{l} S := \sum_{b=0}^{d-1} \left(e^{\frac{2\pi i n}{d}}\right)^b = 1 + r + \dots + r^{d-1} \\ rS = r + \dots + r^d, \quad r = e^{2\pi i n/d} \end{array} \right\} \begin{array}{l} (1-r)S = 1 - r^d \\ = 0. \\ \Rightarrow S = 0. \end{array} \Bigg)$$

Hence we can drop all the n's except those of the form $n = dl$ for some l.

We get $T(m)f = m^{k-1} \sum_{l=0}^{\infty} \sum_{\substack{ad=m \\ a,d}} c(dl)\, d^{-k} \cdot d\, e^{2\pi i l a z}$

$$= \sum_{l=0}^{\infty} \sum_{\substack{ad=m \\ a,d}} c(dl) \left(\frac{m}{d}\right)^{k-1} e^{2\pi i a l z}$$

$$= \sum_{l=0}^{\infty} \sum_{\substack{a|m \\ a>0}} c\left(l \frac{m}{a}\right) a^{k-1} q^{la}$$

<u>Coef of q^0</u> in this sum comes from $l = 0$ and then is equal to $c(0) \sum_{\substack{a|m \\ a>0}} a^{k-1}$

$$= c(0)\, \sigma_{k-1}(m).$$

<u>Coef of q^1</u> comes from $la = 1$, hence $l = a = 1$ and is just $c(m)$

For $n > 2$, coef of q^n comes from terms l, a with $la = n$ and $a | m$ ie $a | n$, and $a | m$ and $c\left(\frac{lm}{a}\right) = c\left(\frac{nm}{a^2}\right)$

Thus coef of q^n is $\sum_{a | (m,n)} c\left(\frac{nm}{a^2}\right) a^{k-1}$

These formulas also show

Cor 6.3 $T_k(m)$ takes $\mathcal{M}_k(\Gamma)$ to $\mathcal{M}_k(\Gamma)$ and $S_k(\Gamma)$ to $S_k(\Gamma)$.

Cor 6-4 Let $f \in \mathcal{M}_k(\Gamma)$, $f = \sum c(n) q^n$, p a prime, and $g = T_p f$.

Then $g = \sum b(n) q^n$ with

$$b(n) = \begin{cases} c(pn) & \text{if } p \nmid n \\ c(pn) + p^{k-1} c(n/p) & \text{if } p \mid n \end{cases}$$

$$T(n) f = n^{k/2-1} \sum_{\substack{\gamma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \\ ad = n \\ b \bmod d}} f|\gamma = n^{k/2-1} \sum_{\substack{ad=n \\ 0 \le b < d}} (ad)^{k/2} d^{-k} f\left(\frac{az+b}{d}\right)$$

$$= n^{-1} \sum_{ad=n} a^k \sum_{0 \le b < d} f\left(\frac{az+b}{d}\right).$$

We'll formally write this as

$$T_n = \frac{1}{n} \sum_{ad=n} a^k \sum_{b \bmod d} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}.$$

The next theorem is at the heart of the multiplicative relations satisfied by the Fourier coefs of eigenfunctions of T_n

Thm 6.5 For any $m, n > 1$ we have

$$T_m T_n = \sum_{d \mid (m,n)} d^{k-1} T_{\frac{mn}{d^2}} \quad (*)$$

Before we give a proof of this Thm let's look at some of its corollaries

Cor 6.6 The Hecke operators commute

ie $T_n T_m = T_m T_n$

Pf. This is immediate from the symmetry in n, m in $(*)$.

Cor 6.7 (a) If $(m,n) = 1$ then

$T_m T_n = T_{mn}$.

(b) $T_{p^r} T_p = T_{p^{r+1}} + p^{k-1} T_{p^{r-1}}$

Pf. (a) Obvious from $(*)$ since $d = (m,n) = 1$

(b) $$T_{p^r} T_p = \sum_{d \mid \underbrace{(p^r, p)}_{p}} d^{k-1} T_{\frac{p^{r+1}}{d^2}}$$

$$= \underbrace{T_{p^{r+1}}}_{d=1} + \underbrace{p^{k-1} T_{p^{r-1}}}_{d=p}.$$

Next we look at an implication of Thm 6.5 and Prop 6.2 on the eigenvalues and F. coefs of an $f \in \mathcal{M}_k(\Gamma)$ which is an eigenfunction $\forall T_m$.

<u>Prop 6.8.</u> Let $f = \sum_1 a_n q^n \in \mathcal{M}_k$ such that $\forall m \geq 1$ $\exists \lambda_m \in \mathbb{C}$ with $T_m f = \lambda_m f$, and f is not a constant function.

Then $a_1 \neq 0$ and $a_m = \lambda_m a_1$

<u>Proof.</u> $T_m f = \sum b_n q^n = \sum \lambda_m a_n q^n = \lambda_m f$

On the other hand Prop 6.1 gives

$$\lambda_m a_n = b_n = \sum_{d \mid (m,n)} d^{k-1} a_{\frac{mn}{d^2}}$$

Let $n=1$. Then $\lambda_m a_1 = b_1 = \sum_{d \mid (m,1)} d^{k-1} a_{\frac{mn}{d^2}}$

$$= a_m$$

$\Rightarrow \boxed{\lambda_m a_1 = a_m}$ Hence $a_1 \neq 0$ since otherwise $a_m = 0$ $\forall m \geq 1$ and $f \equiv a_0$.

<u>Rmk</u> (1) Prop 6.8 says that up to normalization ($a_1 = 1$), eigenvalues of f and fourier coefs of f are equal. If f is a simultaneous eigenform $\forall T_m$.