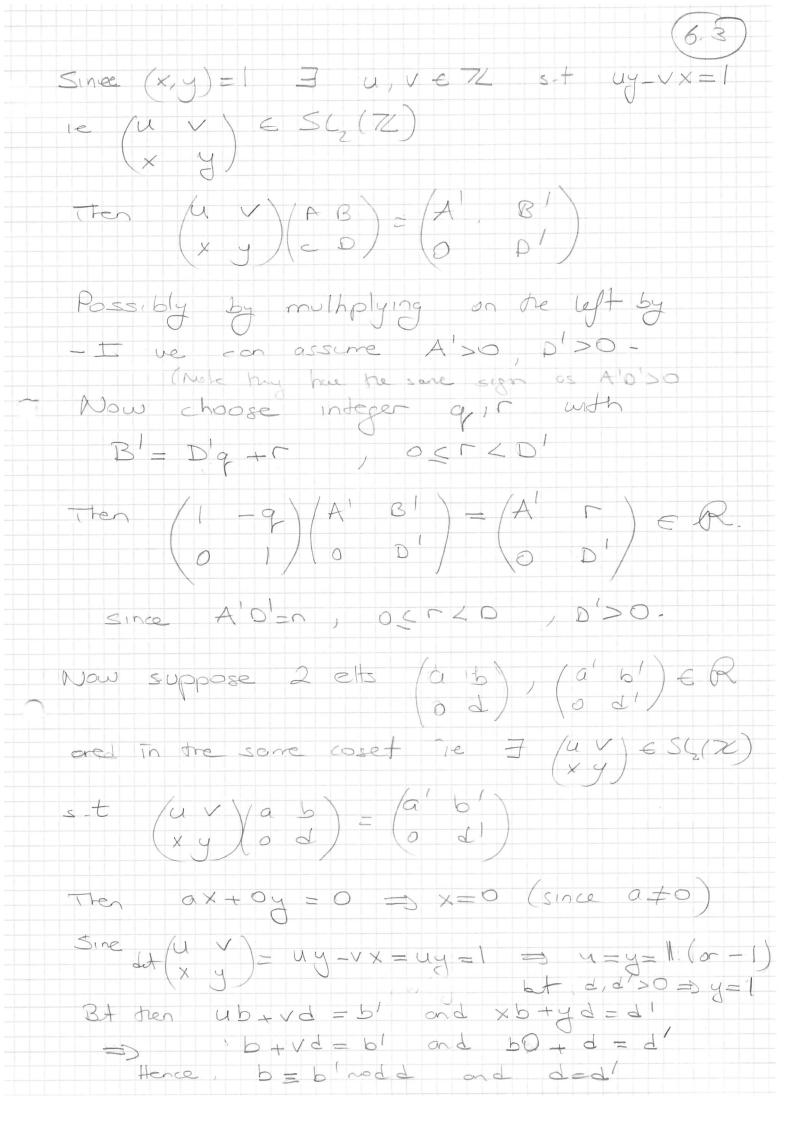
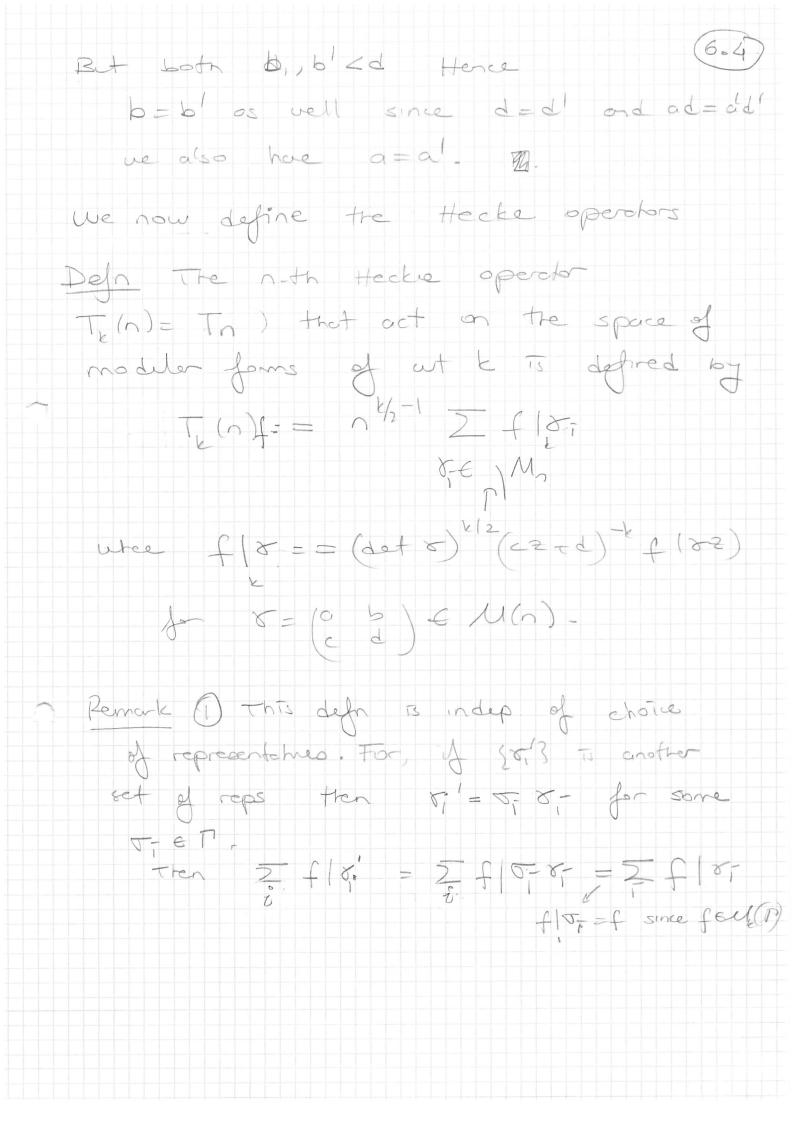
(6-1) §6. Heeke operators We're seen before that for $\Delta(2) = \frac{2}{2} E(n) q^n$ Ramanuter conjectured and Ramanyten confectived and Model proved that Hecke who defred certain operators T(n) acting on the end showed that they form a family of commuting operators that are Hemihan with Petersson inner product. This implies using Linear algebra that the space My has a basis answhat of simultaneous eigenforms VT(n) and the eigenvalues of these forms inhert the multiplication properties of the Heeke operators that act on them. General setup Let $G_1 \leq T \equiv SL_2(Z)$, and GCK, where Kaset of lineor prochonal trensformations (not recessory a gp) with the following properties OG acts on K by m/hplich OKG=GK=K $\Im[K:G] < \infty \ i.e. K= \bigcup G M_{-}$ $M_{-} \in K$ J=1

6-2 Take G = P = 5(z(Z)) $K = \mathcal{M}(n) := \sum m \in \mathcal{M}_{2\times 2}(\mathbb{Z}) \left\{ \det m = n \right\}$ Clearly G acts on K on the right and on the left since y set, gellin) ten det (og) = (deto)(detg) = detg = set (go) The rext lemma gives an explicit set of reps and also proves [U(n): [] [x -Lemma 6.) A set of right caset repr for the action of PSL2(72) on the set Min of of Lires frechal transformetors represented by the set of elemens m(n) is gren by $R = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid ad = n, d > 0 \right\}$ $Te M_{h} = U PSC_{2}(Te) T_{1}, R = M_{h} = M(h)$ $f \in R$ $FSL_{2} SL_{2}$ Prof. let # (A B) & M(n) be gues Choose X, yE The such that AX+ Cy = 0 and (x,y)=1 (For example take x=-c, y=A and durde by ged (A, c))



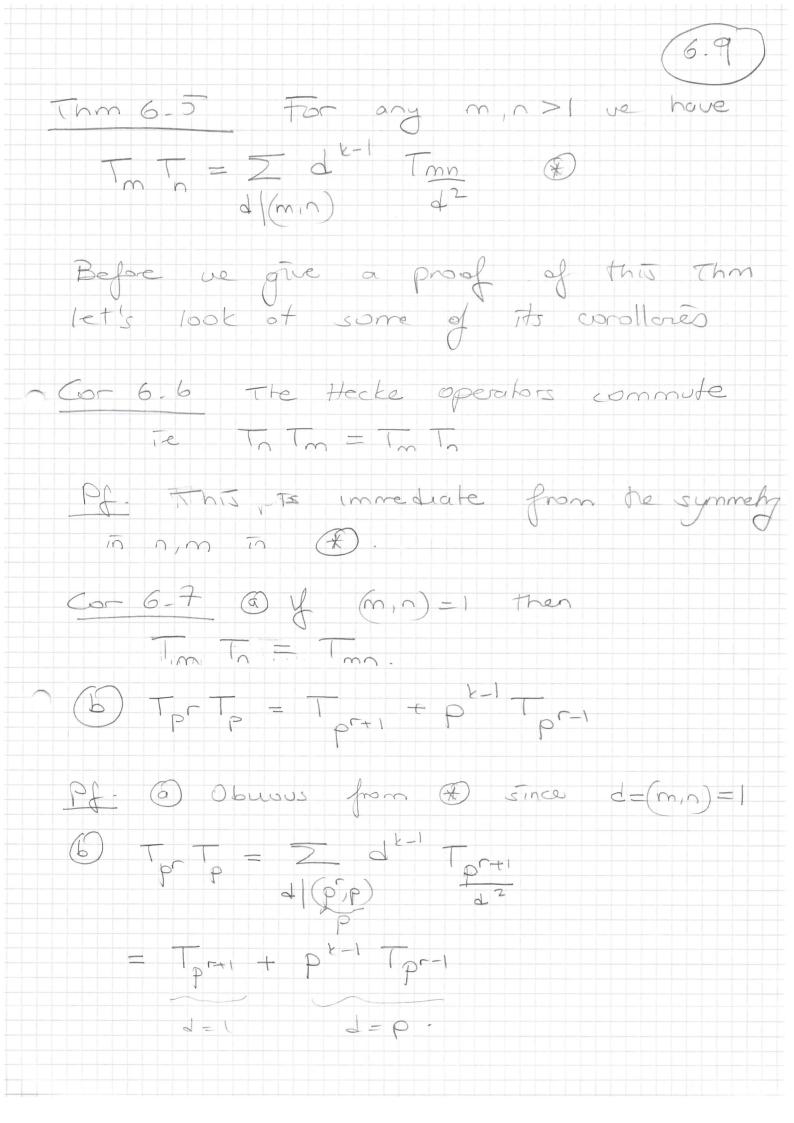


6.5) For, J SET her $g = (T_k(n)f)$ $= \left(\bigcap_{k/2}^{k/2} \sum_{i=1}^{k} f(x_i) \right) \left[\nabla_{i} \right]$ $= n^{k/2-1} \sum_{i=1}^{2} f(v_i \sigma) = n^{k/2-1} \sum_{i=1}^{2} f(v_i - 1)^{i}$ $= f | T_k(n) = q$ Since as of mis over a set of reps (M(n) so doed of =x;5 for a fixed TET. (3) We'll see shortly that in fact $g \in \mathcal{M}_k(\mathcal{P})$ The $T(n) = \mathcal{H}_k(\mathcal{P}) \longrightarrow \mathcal{M}_k(\mathcal{P})$ (ie we show that Trin) also preserves the holomorphicity at no. Using the explicit coset reps from Lemmo 6-1 we will give the action of TE(n) on the Towner cosefs of F.

6-6) $\frac{Prop \ 6.2}{n=0} \ (et \ f(z) = \frac{2}{2} c(n) q^n \in \mathcal{U}_k(n)$ and $q = T_{\ell}(m)f$. Then $q(z) = \sum_{n=0}^{\infty} b(n)q^n$ with $b(n) = \left(\sum_{d \in I} d^{k} c \left(\frac{mn}{d^2} \right) \right) \frac{d}{d} n \ge 1$ $\left(C(0) \sigma_{k-1}(m) \right) \quad i \neq n = 0$ $\frac{p_{nog}}{f_{l}} \cdot \frac{f_{l}}{f_{l}}(m)f = m^{l_{l}-1} \sum_{ad=m} f(a b]$ b mod d 4>0 $= mk(z-1) \sum_{a,b,d} d^{-k} \left(\frac{az+b}{d}\right) mk(z)$ $= m^{k-1} \sum_{a,b,d} -k \sum_{n=0}^{\infty} c(n) e^{2\pi i n} \left(\frac{a^2 + b}{d}\right)$ $= m^{k-1} \sum_{n=0}^{\infty} c(n) \sum_{a,d} f^{k} e^{2\pi i n} \left(\frac{a^{2i}}{d} \right) \sum_{d=0}^{d-1} \frac{2\pi i n}{d} \left(\frac{b}{d} \right)$ $= m^{k-1} \sum_{n=0}^{n=0} a, d \qquad b=0$ = a d = m = d = 1 $= 2\pi i n \left(\frac{b}{d} \right) = \left(\frac{d}{d} + \frac{d}{d} \right) n$ = 0 = 1 = 1 = 1 = 1 = 1 = 1 = 1 = 1



(6-8) These formulas also show $\frac{\text{cor } 6.3}{\text{ond}} \quad T_{k}(m) \quad \text{fakes} \quad \mathcal{U}_{k}(r) \quad \text{for } \mathcal{U}_{k}(r)$ $\text{ond} \quad S_{k}(r) \quad \text{for } S_{k}(r).$ Cor 6-4 let $f \in M_{k}(P)$, $f = Zc(n)q^{2}$ Papme, and g=Tpf Then $g = \sum b(n)g^n$ with $b(n) = (c(pn)) \neq p \neq n$ (c(pn) + pt-1 c(n/p) y pln $T(n)f = n^{k/2-1} \sum_{\substack{f \mid d \\ o \neq b}} f \mid d = n^{k/2-1} \sum_{\substack{d \neq d \\ o \neq b}} d f \mid d = n^{k/2-1} \sum_{\substack{d \neq d \\ o \neq b}$ $z \cap Z a Z f \left(\frac{az_{+}b}{d} \right)$, aden osb&d We'll formally write this as $T_n = \prod_{n=1}^{\infty} \mathbb{Z} \left(\begin{array}{c} a \\ c \\ d \end{array} \right)$ The rext theorem is at the heart of the multiplicative relations so thefred by the Forrer coefs of egenfinchers of Th



Next we look at an implication of the 6-10. and Prop 6-2 on the eigenvalues and Ficoels of many the intervalues and F. coefs of an f & Uk(T) which is an eisenfinchin VTm. Prop 6.8. letuf= Zan grelle sich that Imzl $\exists \lambda_m \in \mathbb{C}$ with $\exists f = \lambda_m f$, and f is not a constant function. Then $a_1 \neq 0$ and $a_m = \lambda_m a_1$ $\frac{Proof}{Tmf} = \frac{Zb_{n}q^{n}}{Tmf} = \frac{Zb$ On the other hand Prop 6-1 gives $\lambda_m a_n = b_n = \sum_{\substack{d \\ d \\ d \\ m,n}} \frac{d^{k-1}}{d^2}$ $| t n = | . Then <math>m_1 = b_1 = \sum_{i=1}^{n} \frac{d^{i-1}}{d_i} \frac{m_1}{d_i^2}$ =) $\left[\lambda_{m} a_{1} = a_{m}\right]$ there $a_{1} \neq 0$ since otherwise $a_{m} = 0$ there 1and $f \equiv a_{a}$ Rmk (AProp 6.8 says that up to normalization (a=1), eignucleus of f and fourer weight of f are equal. If to a 'simultaneous ergenform & Tm.